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FIXED POINT THEOREMS FOR F-WEAKLY CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to present some fixed point theorems for f-weakly maps in complete metric space endowed with a partial order. Also, we give an example for our main theorem.

Keywords: common fixed points, coincidence points, f -weakly contractive mapping, partially ordered metric space.

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1. Introduction

It is well known that the Banach Contraction Principle has been generalized in various direction. Alber and Guerre-Delabrere [1] define weakly contractive maps. Rhoades [15] showed that the result of Alber et al. is also valid in complete metric spaces. Weakly contractive mappings have been used in number of subsequent works to establish various fixed point and common fixed point theorems.

In particular, recent results of Song [16] on f-weakly contractive mappings is among the most general ones. The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [13], and then by Nieto and Lopez [11].

Further results and applications to differential equations in this direction were proved, e.g., in [3, 5, 6, 7, 12]. Results on weakly contractive mappings in such spaces were obtained by Harjani and Sadarangani in [8].

In this paper, we extend results of Song [16] to ordered metric space. Also, we give an example for our main theorem.

2. Preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is called to be weakly contractive [1], [15] if, for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

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where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, +\infty)$ and $\varphi(0) = 0$.
We say that a mapping $T : X \to X$ is f-weakly contractive if, for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)), \tag{1}$$

where $f : X \to X$ is a self–mapping and $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, +\infty)$ and $\varphi(0) = 0$.
If $\varphi(t) = (1-k)t$, $0 < k < 1$, then a f-weakly contractive mapping is called a f–contraction. Note that if $f = I$ and φ is continuous non–decreasing, then the definition of f–weakly contractive mapping is the same as it appeared in [1] and [15]. Further if $f = I$ and $\varphi(t) = (1 - k)t$, $0 < k < 1$, then a f–weakly contractive mapping is called a contraction. Also, note that if $f = I$ and φ is lower semicontinuous from the right then $\psi(t) = t - \varphi(t)$ is upper semicontinuous from the right and condition (1) is replaced by

$$d(Tx, Ty) \leq \psi(d(x, y)). \tag{2}$$

Therefore, f-weakly contractive maps for which φ is lower semicontinuous from the right are of Boyd and Wong [4] type. Further, if we define $k(t) = 1 - \frac{\varphi(t)}{t}$ for $t > 0$ and $k(0) = 0$ together with $f = I$, then condition (1) is replaced by

$$d(Tx, Ty) \leq k(d(x, y))d(x, y). \tag{3}$$

Therefore f-weakly contractive maps are closely related to maps of Reich [14] type, which are also generally researched by Bae [2] and Mizoguchi and Takahashi [10].
Let (X, d) be a metric space, and f, T be two self–mappings on X. The set of fixed points of T we shall denote by $F(T)$. A point $x \in X$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$ and the set of common fixed points of f and T is denoted by $F(f, T)$. The pair f, T is called commuting [9] if $Tfx = fTx$ for all $x \in X$.
Recently, Song [16] extended the Rhoades' theorem in the following way.

Theorem 2.1. *Let (X, d) be a metric space, and $T, f : X \to X$ two self–mappings with $\overline{T(X)} \subseteq f(X)$. Assumed that either $\overline{T(X)}$ or $f(X)$ is complete, and T is f–weakly contractive mapping, then*
(i)$C(T, f) \neq \emptyset$,
(ii)If T and f commute at their coincidence points, then $F(f, T)$ is singleton.

3. Main results

Let $(X, \leq)$ be a partially ordered set and $T, f : X \to X$ be mappings. We say that T and f have the property $\star$ on X, if for each $x \in \overline{T(X)}$ there exists $y \in X$ such that Ty comparable to x and $fy = x$.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X has the following property:*

$$\begin{array}{c} if\ x_n \to x\ is\ a\ sequence\ in\ X\ whose\ consecutive\ terms\ are\ comparable\ then\ there \\ exists\ a\ subsequence\ \{x_{nk}\}\ of\ \{x_n\}\ such\ that\ every\ term\ is\ comparable\ to\ the\ limit\ x. \end{array} \tag{4}$$

Let $T, f : X \to X$ be mappings have the property $\star$ on X, and commute in coincidence points. Assume that

$$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)), \tag{5}$$

for $x, y \in X$ with $fx \leq fy$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semicontinuous function such that φ is positive on $(0, +\infty)$ and $\varphi(0) = 0$. If

$$u \in C(T, f) \Rightarrow \begin{cases} Tu \leq fTu \\ or \\ Tu \geq fTu, \end{cases} \tag{6}$$

then $F(T, f) \neq \emptyset$.

Proof. Using the weakly same image property of T and f, we can construct inductively, starting with arbitrary $x_0 \in X$, a sequence $\{x_n\}$ such that Tx_{n+1} is comparable to Tx_n and $fx_{n+1} = Tx_n$. Since fx_{n+1} and fx_n are comparable for $n = 1, 2, ...$, then from (5), we get

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq d(fx_{n+1}, fx_n) - \varphi(d(fx_{n+1}, fx_n)) = \\ &= d(Tx_n, Tx_{n-1}) - \varphi(d(Tx_n, Tx_{n-1})) \leq \\ &\leq d(Tx_n, Tx_{n-1}). \end{aligned}$$

Therefore, $\{d(Tx_{n+1}, Tx_n)\}$ is decreasing and bounded below. So, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(Tx_{n+1}, Tx_n) = r$. It follows from the lower semicontinuity of φ that

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(d(Tx_{n+1}, Tx_n)).$$

We claim that $r = 0$. In fact, taking upper limit as $n \to \infty$ on two sides of the following inequality

$$\begin{gathered} d(Tx_{n+1}, Tx_n) \leq d(Tx_n, Tx_{n-1}) - \varphi(d(Tx_n, Tx_{n-1})), \\ r \leq r - \liminf_{n\to\infty} \varphi(d(Tx_{n+1}, Tx_n)) \leq r - \varphi(r), \end{gathered}$$

i.e. $\varphi(r) \leq 0$. Thus $\varphi(r) = 0$ by the property of the function φ, furthermore

$$\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r = 0. \tag{7}$$

Next we show that $\{Tx_n\}$ is Cauchy sequence. Suppose not, then there exists a real number $c > 0$, for any $N \in \mathbb{N}$, there exists $m_k \geq n_k \geq N$ such that

$$d(Tx_{m_k}, Tx_{n_k}) \geq c. \tag{8}$$

Furthermore, assume that m_k is the smallest number greater than n_k for which Eq. (8) holds, that is $d(Tx_{m_k-1}, Tx_{n_k}) < c$. Using Eq. (8), we get

$$\begin{aligned} c \leq d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k}) < \\ &< d(Tx_{m_k}, Tx_{m_k-1}) + c \to c (k \to \infty). \end{aligned}$$

Hence $\lim\limits_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) = c$, moreover $\varphi(c) \leq \liminf\limits_{k\to\infty} \varphi(d(Tx_{m_k}, Tx_{n_k}))$. On the other hand, noting Eq. (5),

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Tx_{m_k+1}) + d(Tx_{m_k+1}, Tx_{n_k+1}) + d(Tx_{n_k+1}, Tx_{n_k}) \leq \\ &\leq d(fx_{m_k+1}, fx_{n_k+1}) - \varphi(d(fx_{m_k+1}, fx_{n_k+1})) + \\ &+ d(Tx_{m_k}, Tx_{m_k+1}) + d(Tx_{n_k+1}, Tx_{n_k}) = \\ &= d(Tx_{m_k}, Tx_{n_k}) - \varphi(d(Tx_{m_k}, Tx_{n_k})) + \\ &+ d(Tx_{m_k}, Tx_{m_k+1}) + d(Tx_{n_k+1}, Tx_{n_k}). \end{aligned}$$

Taking upper limit on two sides of above inequality as $k \to \infty$ along with Eq. (3.2), we have

$$c \leq c - \liminf_{k \to \to \infty} \varphi(d(Tx_{m_k}, Tx_{n_k})) \leq c - \varphi(c).$$

Thus $\varphi(c) \leq 0$, that is $c = 0$, which is a contradiction to $c > 0$. Therefore the sequence $\{Tx_n\}$ is Cauchy sequence. Since X is a complete metric space then there exists $z \in X$ such that $Tx_n \to z$. It follows from the property $\star$ of T and f, there exists $u \in X$ such that Tu and z are comparable and $fu = z$.

Since $\{Tx_n\}$ is a sequence with comparable consecutive terms then by condition (4), there exists subsequence $\{Tx_{nk}\}$ consisting of terms which are comparable to the limit z. Hence, for $k \in \mathbb{N}$

$$\begin{aligned} d(Tx_{n_k+1}, Tu) &\leq d(fx_{n_k+1}, fu) - \varphi(d(fx_{n_k+1}, fu)) = \\ &= d(Tx_{n_k}, z) - \varphi(d(Tx_{n_k}, z)). \end{aligned}$$

Taking upper limit as $k \to \infty$, we obtain that $Tx_{n_k+1} \to Tu$, since $Tx_{n_k+1} \to z$, then $Tu = z = fu$. Since $u \in C(T,f)$ and T is commuted with f in u, we have

$$Tz = Tfu = fTu = fz.$$

Also, by condition (6), we obtain that $z = Tu \leq fTu = fz$ or $z = Tu \geq fTu = fz$, hence we get

$$\begin{aligned} d(Tz, z) = d(Tz, Tu) &\leq d(fz, fu) - \varphi(d(fz, fu)) = \\ &= d(fz, z) - \varphi(d(fz, z)) = \\ &= d(Tz, z) - \varphi(d(Tz, z)). \end{aligned}$$

Therefore $\varphi(d(Tz, z)) = 0$. This means that $z \in F(T, f)$. ☐

Corollary 3.1. *If there exists another $v \in X$ such that $v \in F(T, f)$ and v comparable to z then*

$$d(z, v) = d(Tz, Tv) \leq d(fz, fv) - \varphi(d(fz, fv)) = d(z, v) - \varphi(d(z, v)).$$

Hence $z = v$. This shows that $F(T, f)$ is singleton.

Example 3.1. *Let $X = [0,1] \times [0,1]$, and consider the usual order*

$$(x, y) \leq (u, v) \Leftrightarrow x \leq u, y \leq v.$$

Then $(X, \leq)$ is a partially ordered set. Besides, (X, d) is a complete metric space, considering d the Euclidean distance. Let $T, f : X \to X$ be given by $f(x, y) = (\frac{x}{2}, \frac{y}{2})$ and $T(x, y) = (\frac{x^2}{8}, \frac{y^2}{8})$, for all $(x, y) \in X$, and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$.
Now, we check that hypotheses in Theorem (3.1) is satisfied. The mapping T , f have condition (5), since for $(x, y) \leq (u, v)$,

$$d(f(x,y), f(u,v)) - \varphi((d(f(x,y), f(u,v))) = \frac{1}{4}\sqrt{(x-u)^2 + (y-v)^2} = \frac{1}{4} d((x,y),(u,v))$$

and

$$\begin{aligned} d(T(x,y), T(u,v)) = \frac{1}{8}\sqrt{(x-u)^2(x+u)^2 + (y-v)^2(y+v)^2} &\leq \frac{1}{4}\sqrt{(x-u)^2 + (y-v)^2} = \\ &= d(f(x,y), f(u,v)) - \varphi((d(f(x,y), f(u,v))). \end{aligned}$$

It is obvious that $T(X) = X$. For $(x, y) \in X$, put $(u, v) = (\frac{x^2}{4}, \frac{y^2}{4})$, we obtain

$$T(u, v) = (\frac{x^4}{128}, \frac{y^4}{128}) \leq (\frac{x^2}{8}, \frac{y^2}{8}) = T(x, y)$$

and

$$f(u,v) = (\frac{u}{2}, \frac{v}{2}) = (\frac{x^2}{8}, \frac{y^2}{8}) = T(x,y).$$

This implies that T , f have the property $\star$. It is easy to see that X has the condition (4). It is obvious that $C(T,f) = F(T,f) = \{0\}$.

Corollary 3.2. *Let $(X, \leq)$ be a partially ordered set and there exists metric d in X, such that (X,d) is a complete metric space. Assume that X satisfies in condition (4). Let $T, f : X \to X$ be a mappings having the property $\star$ on X and commute in coincidence points. Assume that*

$$d(Tx,Ty) \leq \psi(d(fx,fy)), \tag{9}$$

for $x, y \in X$ with $fx \leq fy$, where $\psi : [0,+\infty) \to [0,+\infty)$ is upper semicontinuous such that $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$. If for all $x \in C(T,f)$, $Tx \leq fTx$ or $Tx \geq fTx$, then $F(T,f) \neq \emptyset$.

Proof. Set $\varphi(t) = t - \psi(t)$, then Eq.(9) implies that

$$d(Tx,Ty) \leq d(fx,fy) - \varphi(d(fx,fy)),$$

for $x, y \in X$ with $fx \leq fy$, and also $\varphi : [0,+\infty) \to [0,+\infty)$ is lower semicontinuous function and $\varphi(t) = 0$ if and only if $t = 0$. The result follows from Theorem (3.1). □

Corollary 3.3. *Let $(X, \leq)$ be a partially ordered set and there exists metric d in X such that (X,d) is a complete metric space. Assume that X satisfies in condition (4). Let $T, f : X \to X$ be a mappings having the property $\star$ on X and commute in coincidence points. Assume that*

$$d(Tx,Ty) \leq \alpha(d(fx,fy))d(fx,fy), \tag{10}$$

for $x, y \in X$ with $fx \leq fy$, where $\alpha : (0,+\infty) \to (0,1)$ is upper semicontinuous. If for all $x \in C(T,f)$, $Tx \leq fTx$ or $Tx \geq fTx$, then $F(T,f) \neq \emptyset$.

Proof. Set $\varphi(t) = (1-\alpha(t))t$, then Eq. (10) implies that

$$d(Tx,Ty) \leq d(fx,fy) - \varphi(d(fx,fy)),$$

for $x, y \in X$ with $fx \leq fy$, and moreover $\varphi : [0,+\infty) \to [0,+\infty)$ is lower semicontinuous function such that $\varphi(t) = 0$ if and only if $t = 0$. It follows from Theorem (3.1) that we reach our aim. □

Let $f = I$ the identity operator, and $\varphi(t) = (1-k)t$ for a constant k with $0 < k < 1$, then we easily obtain the following result.

Corollary 3.4. *Let $(X, \leq)$ be a partially ordered set and there exists a complete metric d in X. Let $T : X \to X$ be a nondecreasing mapping such that*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y))$$

for $x \geq y$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Assume that X is such that:
if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.
If there exist $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof. Since $x_0 \leq Tx_0$ and T is nondecreasing function, we obtain that

$$x_0 \leq Tx_0 = x_1 \leq T^2x_0 = x_3 \leq ... \leq T^nx_0 = x_{n+1} \leq ....$$

Following the proof of Theorem (3.1), we get the result. □

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